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Negative dimensional integration: "lab testing" at two loops

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Abstract: Negative dimensional integration method (NDIM) is a technique to deal with D-dimensional Feynman loop integrals. Since most of the physical quantities in perturbative Quantum Field Theory (pQFT) require the ability of solving them, the quicker and easier the method to evaluate them the better. The NDIM is a novel and promising technique, *ipso facto* requiring that we put it to test in different contexts and situations and compare the results it yields with those that we already know by other well-established methods. It is in this perspective that we consider here the calculation of an on-shell two-loop three point function in a massless theory. Surprisingly this approach provides twelve non-trivial results in terms of double power series. More astonishing than this is the fact that we can show these twelve solutions to be different representations for the same well-known single result obtained via other methods. It really comes to us as a surprise that the solution for the particular integral we are dealing with is twelvefold degenerate.

Keywords: Renormalization Regularization and Renormalons, Space-Time Symmetries.

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Contents

1 Introduction

Negative dimensional integration method (NDIM) was first devised by Halliday *et al.* [1, 2] with the aim of dealing with Feynman loop integrals in quantum field theory (QFT). It combines the powerful concepts of analytic continuation, translational invariance and dimensional regularization [3, 4, 5, 6] (DREG) in such a way that the intricate positive dimensional integration is transformed into negative dimensional integration of polynomial type. In practical terms, what one needs to do is to solve systems of linear algebraic equations and gaussian-like integrals.

Before plunging deeply in the mire of it, let us briefly make some comments. First of all, what do we mean by *negative dimensions*? Obviously, at their face value they can only be fictional. Nevertheless, in a stretch of our imagination, if we allow ourselves just the possibility of their existence, maybe we can bring into fruition something ever undreamed of before. So, to begin with, let us mirror our reasoning with that behind DREG. There, the all important concept is the parameter D, the space-time dimension, allowed to assume complex values. However, this in no way means that we want or will even define operations like scalar product [4, 5] in some general D to have reality. The reason is quite simple: Physics — and even the common sense — tells us that D is a positive integer number. We keep our eyes in this fact. One may think that this way of expressing the result of a given Feynman integral in an arbitrary space-time dimension is very elegant indeed, but at the end of the day, one always has to look into the real physical world, that is, that of positive integer D. Actually, in the whole process of DREG it is important to keep D arbitrary because the analyticity properties depend on the space-time dimension and, of course, any singularities do so depend on it too. Wilson [5] noted, very early, that the so-called integration in D-dimensions is not in fact real literally. It only seems to be. It only behaves like it is.

So, we too define an integral in D-dimensions by means of an analytic function and then work with general D, negative values included. We will not attempt to make any sense of it or speculate about the $D < 0$ "world". What we do is just allow for it without trying to see how real — or how fictional, for that matter — it is, before doing the analytic continuation back to $D > 0$. In other words, we are not concerned with seeking any meaning for a negative dimensional world nor seeking any new Physics.

Let us sketch the methodology proper. The idea is quite simple: we analytically continue the Feynman integral we want to evaluate into $D < 0$ and solve it there. Then, the result we get there we bring it back into the realm of $D > 0$, by another analytic continuation. As we shall see, the whole procedure is much easier to do compared to positive dimensional techniques. Up to now, for all the Feynman integrals we have calculated using NDIM, the results agree with the ones calculated in the positive dimensional regime [7, 8, 9]. This includes even some light-cone gauge loop integrals [10], which knowingly are harder to solve with other approaches.

The outline of our paper is as follows: in Sec. 2 we solve explicitly a two-loop Feynman integral entering the two-loop radiative correction to the massless triangle diagram using the negative–D approach. Then, we show how to analytically continue the result to positive D and give the result for two on-shell external legs. In Sec. 3 we conclude this work commenting on some new results we have for massive one-loop and off-shell two-loop Feynman integrals. Finally, in the Appendix we list all the other remaining degenerate solutions, for the sake of completeness.

2 On-Shell Two-loop Vertex

As we have mentioned earlier, our work here is done within the perpective of checking NDIM methodology for D-dimensional Feynman integrals. Therefore, we have chosen as our "lab test" for it, the evaluation of an integral pertinent to the Feynman diagram of Fig.1. It is a two-loop graph with four particles in the intermediate states, i.e., containing four propagators. For simplicity we take the massless case and in order to compare our result with the one already known in the literature, we take a particular limit of two on-shell external legs.

Since ours is a choice example, we cannot become too excited about NDIM and its seemingly simplicity, yet one can convince himself that the task of solving Feynman loop integrals is quite easy in this approach (at least in principle). Of course, we can envisage and even anticipate some technical difficulties in other contexts, which is inherent to the method, such as the necessity of dealing with multi-indexed summations.

NDIM is implemented with a few simple steps: Firstly, we calculate gaussian or gaussian-type integrals[1], which are not difficult to handle, and some books on QFT even list them in tables. Then, one makes an expansion in Taylor series of the result obtained, and compare it with the expansion in Taylor series of the original gaussian or gaussian-type integral. Comparison term by term of both series then yield a system of linear

[1] In D-dimensional momentum space.

algebraic equations, with constraints arising from intermediate multinomial expansions. NDIM requires that we solve this system. The principle therefore is quite simple! Let us then take a practical example and see how it works.

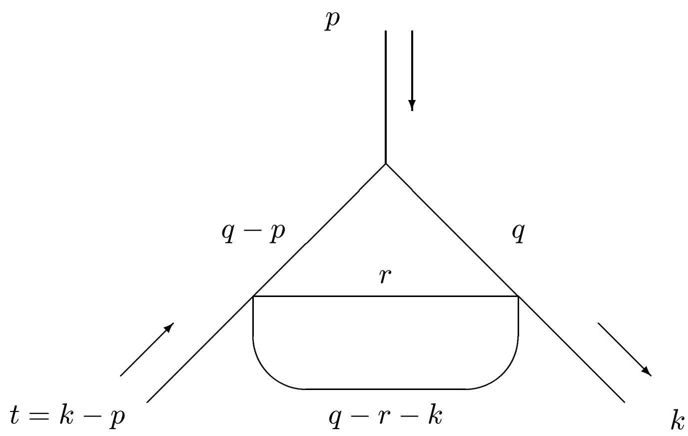

Figure 1: Two-loop three point vertex.

Let our launching-pad gaussian-like double integral be

$$I=\int\int d^Dr\, d^Dq\ \ \exp\left[-\alpha q^2-\beta(q-p)^2-\gamma r^2-\omega(q-r-k)^2\right]. \tag{2.1}$$

This clearly is a pertinent integral to the diagram of Fig.1. For reasons of simplicity and future comparison, let us consider that two of the external particles are real, i.e., let them be on-shell, namely, $k^2=t^2=0$.

Completing the square in the variable q we can carry out the first integration and get

$$\begin{aligned} I &= \left(\frac{\pi}{\lambda}\right)^{\frac{1}{2}D} e^{-\beta p^2+\frac{\beta^2p^2}{\lambda}+\frac{2\beta\omega pk}{\lambda}}\int d^Dr\exp\left(-\theta r^2-2\omega rk\right) \\ &\qquad\times\exp\left[\frac{1}{\lambda}\left(\omega^2r^2+2\beta\omega pr+2\omega^2rk\right)\right], \end{aligned} \tag{2.2}$$

where $\theta=\gamma+\omega$ and $\lambda=\alpha+\beta+\omega$. Following the same procedure we perform the remaining integration, remembering that $t=k-p$ and that because of the on-shell condition $t^2=k^2=0$, $p^2=2pk$:

$$I=\left(\frac{\pi^2}{\phi}\right)^{\frac{1}{2}D}\exp\left[\frac{-1}{\phi}(\alpha\beta\gamma+\alpha\beta\omega)p^2\right], \tag{2.3}$$

where $\phi=\alpha\gamma+\alpha\omega+\beta\gamma+\beta\omega+\gamma\omega$. Expanding the exponential in Taylor series and using the multinomial expansion in ϕ, we get

$$\begin{aligned} I &= \pi^D\sum_{\{n_i=0\}}^{\infty}\frac{(-p^2)^{n_1+n_2}(-n_1-n_2-\frac{1}{2}D)!}{n_1!n_2!n_3!n_4!n_5!n_6!n_7!} \\ &\qquad\times\alpha^{n_1+n_2+n_3+n_4}\beta^{n_1+n_2+n_5+n_6}\gamma^{n_1+n_3+n_5+n_7}\omega^{n_2+n_4+n_6+n_7}, \end{aligned} \tag{2.4}$$

where the summation indices must satisfy the constraint $-n_1-n_2-\frac{1}{2}D=n_3+n_4+n_5+n_6+n_7$ which is the condition imposed by the multinomial expansion.

Now comes the trick [1, 7, 8, 9, 10] of negative dimensions: expand the integral (2.1) in Taylor series,

$$\begin{aligned} I &= \sum_{i,j,l,m=0}^{\infty} \frac{(-1)^{i+j+l+m}\alpha^i\beta^j\gamma^l\omega^m}{i!j!l!m!} \\ &\quad \times \int\int d^Dq\; d^Dr\; (q^2)^i \left[(q-p)^2\right]^j (r^2)^l \left[(r-q+k)^2\right]^m , \end{aligned} \tag{2.5}$$

and define

$$J_{NDIM} = \int d^Dq \int d^Dr\; (q^2)^i \left[(q-p)^2\right]^j (r^2)^l \left[(r-q+k)^2\right]^m . \tag{2.6}$$

We already note that this would be exactly the Feynman integral needed to be evaluated for the diagram in Fig.1 if it were not for the positive exponents i, j, l and m.

Comparing now (2.4) and (2.5) we conclude that

$$\begin{aligned} J_{NDIM} &= \frac{\pi^D g(i,j,l,m)}{(-1)^{i+j+l+m}} \sum_{\{n_i=0\}}^{\infty} \frac{(-p^2)^{n_1+n_2}\Gamma(1-n_1-n_2-\frac{1}{2}D)}{n_1!n_2!n_3!n_4!n_5!n_6!n_7!} \\ &\quad \times \delta_{n_1+n_2+n_3+n_4,i}\delta_{n_1+n_2+n_5+n_6,j}\delta_{n_1+n_3+n_5+n_7,l}\delta_{n_2+n_4+n_6+n_7,m}, \end{aligned} \tag{2.7}$$

where

$$g(i,j,l,m) = \Gamma(1+i)\Gamma(1+j)\Gamma(1+l)\Gamma(1+m).$$

We can immediately see that there is a seven index summation[2] and five equations linking them (remembering that one comes from the constraint imposed by the multinomial expansion). Therefore, altogether the system can be solved in twenty-one different ways with two remaining series[3]. Nine of them are trivial solutions, which present no interest at all. The remaining twelve we must solve one by one. In principle we do not know whether these are equivalent or not [7, 11, 12].

A little bit of algebraic rearrangement yields

$$\begin{aligned} J_{NDIM} &= (-\pi)^D (p^2)^\sigma g(i,j,l,m)\Gamma(1-\sigma-\frac{1}{2}D) \\ &\quad \times \sum_{\{n_i=0\}}^{\infty} \frac{\delta_{n_1+n_2+n_3+n_4,i}\delta_{n_1+n_2+n_5+n_6,j}\delta_{n_1+n_3+n_5+n_7,l}\delta_{n_2+n_4+n_6+n_7,m}}{n_1!n_2!n_3!n_4!n_5!n_6!n_7!}, \end{aligned} \tag{2.8}$$

where we have defined $\sigma = i+j+l+m+D$.

It is a simple matter to write down a computer program that solves the system exactly in all its twenty-one different ways. One of the solutions, namely, that with remaining sum indices n_2 and n_6 is

$$\begin{aligned} S_1 &= \frac{(-\pi)^D (p^2)^\sigma \Gamma(1+i)\Gamma(1+j)\Gamma(1+l)\Gamma(1+m)\Gamma(1-\sigma-\frac{1}{2}D)}{\Gamma(1+\sigma)\Gamma(1-j-m-\frac{1}{2}D)\Gamma(1-l-\frac{1}{2}D)\Gamma(1+j-\sigma)} \\ &\quad \times \frac{1}{\Gamma(1+l+m+\frac{1}{2}D)} \sum_{n_2,n_6=0}^{\infty} \frac{(-\sigma|n_2)(\frac{1}{2}D+l|n_2+n_6)(\sigma-j|n_6)}{n_2!n_6!(1-j-m-\frac{1}{2}D|n_2+n_6)}, \end{aligned} \tag{2.9}$$

[2] I.e., a "heptaple" series.

[3] The remnant double series.

where

$$(a|b) \equiv (a)_b = \frac{\Gamma(a+b)}{\Gamma(a)}$$

is the Pochhammer symbol [13, 14, 15] and we use one of its properties, i.e.

$$(a|-k) = \frac{(-1)^k}{(1-a|k)}, \tag{2.10}$$

within the double series. Note that the sum above can be rewritten in terms of hypergeometric functions ${}_2F_1$ if we use another property of the Pochhammer symbol, i.e., $(a|b+c) = (a+b|c)(a|b)$. Now, a hypergeometric function ${}_2F_1$ with unit argument can, within certain constraints in its arguments, be summed [13, 14, 15],

$${}_2F_1(a,b;c|1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{2.11}$$

Rearranging first the n_6 sum we have

$$\begin{aligned} S_1 &= \frac{(-\pi)^D (p^2)^\sigma \Gamma(1+i)\Gamma(1+j)\Gamma(1+l)\Gamma(1+m)\Gamma(1-\sigma-\frac{1}{2}D)}{\Gamma(1+\sigma)\Gamma(1-j-m-\frac{1}{2}D)\Gamma(1-l-\frac{1}{2}D)\Gamma(1+j-\sigma)} \\ &\quad \times \sum_{n_2=0}^{\infty} \frac{(-\sigma|n_2)(\frac{1}{2}D+l|n_2)}{n_2!(1-j-m-\frac{1}{2}D|n_2)} \sum_{n_6=0}^{\infty} \frac{(\sigma-j|n_6)(\frac{1}{2}D+l+n_2|n_6)}{n_6!(1-j-m-\frac{1}{2}D+n_2|n_6)} \\ &\quad \times \frac{1}{\Gamma(1+l+m+\frac{1}{2}D)}, \end{aligned} \tag{2.12}$$

where the second series is by definition the gaussian hypergeometric function, ${}_2F_1$. Using (2.11) we can sum it and then sum also the series in n_2, to get

$$\begin{aligned} S_1 &= \frac{(-\pi)^D (p^2)^\sigma \Gamma(1+i)\Gamma(1+j)\Gamma(1+l)\Gamma(1+m)\Gamma(1-\sigma-\frac{1}{2}D)}{\Gamma(1+\sigma)\Gamma(1-l-\frac{1}{2}D)\Gamma(1+j-\sigma)\Gamma(1+l+m+\frac{1}{2}D)} \\ &\quad \times \frac{\Gamma(1-l-m-D)}{\Gamma(1+i-\sigma)\Gamma(1-m-\frac{1}{2}D)}. \end{aligned} \tag{2.13}$$

Now, grouping the gamma functions in convenient Pochhammer symbols and using (2.10) we analytic continue to negative values of the exponents $i, j, l,$ and m and we go back to positive D to get

$$\begin{aligned} S_1^{AC} &= \pi^D (p^2)^\sigma (-i|\sigma)(-j|\sigma)(-l|-m-\frac{1}{2}D)(-m|\sigma-i-j-\frac{1}{2}D) \\ &\quad \times(\sigma+\frac{1}{2}D|-2\sigma-\frac{1}{2}D)(D+l+m|-l-\frac{1}{2}D). \end{aligned} \tag{2.14}$$

This is the general result, in Euclidean space, for the Feynman graph of Fig.1.

In the important particular case when $i=j=l=m=-1$, the result is

$$S_1^{AC} = \frac{\pi^D (p^2)^{D-4} \Gamma^2(D-3)\Gamma^2(\frac{1}{2}D-1)\Gamma(2-\frac{1}{2}D)\Gamma(4-D)}{\Gamma(D-2)\Gamma(\frac{3}{2}D-4)}, \tag{2.15}$$

which is the well-known result in D-dimensions [16].

We ask immediately: what is the result that the other solutions provide? The answer is as surprising as it could be: the result is the same. All the twelve non-trivial solutions, even distinct, give the same result. It is an amazing feature revealed by NDIM.

Just to check on this, let us consider another solution, for instance, the one where n_2 and n_5 are the remaining sum indices,

$$\begin{aligned} S_2 &= \frac{(-\pi)^D (p^2)^\sigma \Gamma(1+i)\Gamma(1+j)\Gamma(1+l)\Gamma(1+m)\Gamma(1-\sigma-\frac{1}{2}D)}{\Gamma(1+\sigma)\Gamma(1-\sigma-m-\frac{1}{2}D)\Gamma(1+j-\sigma)\Gamma(1+l+m+\frac{1}{2}D)} \\ &\times \frac{1}{\Gamma(1+i+m+\frac{1}{2}D)} \sum_{n_2,n_5=0}^{\infty} \frac{(-1)^{n_2+n_5}(-\sigma|n_2)(-j+\sigma|n_5)}{n_2!n_5!(1-\sigma-m-\frac{1}{2}D|n_2-n_5)} \\ &\times \frac{1}{(1+i+m+\frac{1}{2}D|n_5-n_2)}. \end{aligned} \quad (2.16)$$

As in the previous case we can sum both series. Using (2.10) one can rewrite the n_5 series and identify it as a ${}_2F_1$ function,

$$\begin{aligned} S_2 &= (-\pi)^D (p^2)^\sigma P_2(i,j,l,m;D) \sum_{n_2=0}^{\infty} \frac{(-1)^{n_2}(-\sigma|n_2)}{n_2!(1-\sigma-m-\frac{1}{2}D|n_2)} \\ &\times \frac{1}{(1+i+m+\frac{1}{2}D|-n_2)} \sum_{n_5=0}^{\infty} \frac{(\sigma-j|n_5)(\sigma+m+\frac{1}{2}D-n_2|n_5)}{n_5!(1+i+m+\frac{1}{2}D-n_2|n_5)}, \end{aligned} \quad (2.17)$$

where

$$\begin{aligned} P_2(i,j,l,m;D) &= \frac{\Gamma(1+i)\Gamma(1+j)\Gamma(1+l)\Gamma(1+m)\Gamma(1-\sigma-\frac{1}{2}D)}{\Gamma(1+\sigma)\Gamma(1-\sigma-m-\frac{1}{2}D)\Gamma(1+j-\sigma)} \\ &\times \frac{1}{\Gamma(1+l+m+\frac{1}{2}D)\Gamma(1+i+m+\frac{1}{2}D)}. \end{aligned}$$

Summing the n_5 series with formula (2.11) and using again (2.10) to rewrite the n_2 series, we get

$$\begin{aligned} S_2 &= (-\pi)^D (p^2)^\sigma P_2(i,j,l,m;D) \frac{\Gamma(1+i+m+\frac{1}{2}D)\Gamma(1+i+j-2\sigma)}{\Gamma(1+i-\sigma)\Gamma(1-l-\frac{1}{2}D)} \\ &\times \sum_{n_2=0}^{\infty} \frac{(-\sigma|n_2)(\frac{1}{2}D+l|n_2)}{n_2!(1-\sigma-m-\frac{1}{2}D|n_2)}. \end{aligned} \quad (2.18)$$

This series is by definition a summable ${}_2F_1$ function; with the help of eq. (2.11), we get the expression (2.13) which leads to the correct result (2.15).

The reader can prove, following the same procedure, that all the twelve solutions provide the correct result. The question that arises is: Why is this so? We have no answer to this puzzle at the moment and can only conjecture that maybe if the remaining series has unity argument and is summable then the result will be degenerate. Of course, further research is necessary in order to prove or disprove this conjecture.

3 Conclusion

Our two-loop "lab testing" for NDIM approach to calculate Feynman integrals has revealed some very interesting features of the method. The methodology is quite simple: solving gaussian integrals and systems of linear algebraic equations. NDIM yielded twelve non-trivial solutions which give the same result, eq. (2.14), for the general case, i.e., D-dimensions and arbitrary exponents of propagators. This work encourages us to tackle a more difficult task: the calculation of massive four point one-loop integrals [7, 11] and off-shell two-loop Feynman graphs [12]. Work in this line is in progress, and we have already obtained some more encouraging results. For the latter, for example, a new surprise with NDIM yielding twenty-one distinct (and new!) results, some of them in terms of Appel's [17] hypergeometric functions F_4 which are simpler than the usual dilogarithms [18]. These new results will be the subject addressed in our shortly forthcoming paper.

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A The Remaining Solutions

For the sake of completeness, in this appendix we list the remaining ten non-trivial solutions since in the article proper we have explicitly shown only two of them, and that these two give the correct result. All of these can be summed and analytically continued to positive D with the same ideas we used in section 2.

$$\begin{aligned} S_3 &= \frac{(-\pi)^D(p^2)^\sigma\Gamma(1+i)\Gamma(1+j)\Gamma(1+l)\Gamma(1+m)\Gamma(1-\sigma-\frac{1}{2}D)}{\Gamma(1+\sigma+l+\frac{1}{2}D)\Gamma(1-l-\frac{1}{2}D)\Gamma(1+i-\sigma)\Gamma(1+j-\sigma)} \\ &\quad\times\frac{1}{\Gamma(1+l+m+\frac{1}{2}D)}\sum_{n_4,n_6=0}^{\infty}\frac{(-j+\sigma|n_6)(\frac{1}{2}D+l|n_4+n_6)(-i+\sigma|n_4)}{n_4!n_6!(1+l+\frac{1}{2}D+\sigma|n_4+n_6)}, \end{aligned} \tag{A.1}$$

$$\begin{aligned} S_4 &= \frac{(-\pi)^D(p^2)^\sigma\Gamma(1+i)\Gamma(1+j)\Gamma(1+l)\Gamma(1+m)\Gamma(1-\sigma-\frac{1}{2}D)}{\Gamma(1+j+l+\frac{1}{2}D)\Gamma(1+i+m+\frac{1}{2}D)\Gamma(1+j-\sigma)\Gamma(1+l+m+\frac{1}{2}D)} \\ &\quad\times\frac{1}{\Gamma(1+i-\sigma)}\sum_{n_4,n_5=0}^{\infty}\frac{(-1)^{n_4+n_5}(-i+\sigma|n_4)(-j+\sigma|n_5)}{n_4!n_5!(1+j+l+\frac{1}{2}D|n_4-n_5)(1+i+m+\frac{1}{2}D|n_5-n_4)}, \end{aligned} \tag{A.2}$$

$$\begin{aligned} S_5 &= \frac{(-\pi)^D(p^2)^\sigma\Gamma(1+i)\Gamma(1+j)\Gamma(1+l)\Gamma(1+m)\Gamma(1-\sigma-\frac{1}{2}D)}{\Gamma(1+i+l+\frac{1}{2}D)\Gamma(1+j+m+\frac{1}{2}D)\Gamma(1+j-\sigma)\Gamma(1+l+m+\frac{1}{2}D)} \\ &\quad\times\frac{1}{\Gamma(1+i-\sigma)}\sum_{n_3,n_6=0}^{\infty}\frac{(-1)^{n_3+n_6}(-i+\sigma|n_3)(-j+\sigma|n_6)}{n_3!n_6!(1+i+l+\frac{1}{2}D|n_6-n_3)(1+j+m+\frac{1}{2}D|n_3-n_6)}, \end{aligned} \tag{A.3}$$

$$S_6 = \frac{(-\pi)^D (p^2)^\sigma \Gamma(1+i)\Gamma(1+j)\Gamma(1+l)\Gamma(1+m)\Gamma(1-\sigma-\frac{1}{2}D)}{\Gamma(1+\sigma+m+\frac{1}{2}D)\Gamma(1-m-\frac{1}{2}D)\Gamma(1+i-\sigma)\Gamma(1+j-\sigma)} \times \frac{1}{\Gamma(1+l+m+\frac{1}{2}D)} \sum_{n_3,n_5=0}^{\infty} \frac{(-i+\sigma|n_3)(-j+\sigma|n_5)(\frac{1}{2}D+m|n_3+n_5)}{n_3!n_5!(1+m+\frac{1}{2}D+\sigma|n_3+n_5)}, \tag{A.4}$$

$$S_7 = \frac{(-\pi)^D (p^2)^\sigma \Gamma(1+i)\Gamma(1+j)\Gamma(1+l)\Gamma(1+m)\Gamma(1-\sigma-\frac{1}{2}D)}{\Gamma(1+\sigma)\Gamma(1-i-m-\frac{1}{2}D)\Gamma(1+i-\sigma)\Gamma(1-l-\frac{1}{2}D)} \times \frac{1}{\Gamma(1+l+m+\frac{1}{2}D)} \sum_{n_2,n_4=0}^{\infty} \frac{(-\sigma|n_2)(-i+\sigma|n_4)(\frac{1}{2}D+l|n_2+n_4)}{n_2!n_4!(1-i-m-\frac{1}{2}D|n_2+n_4)}, \tag{A.5}$$

$$S_8 = \frac{(-\pi)^D (p^2)^\sigma \Gamma(1+i)\Gamma(1+j)\Gamma(1+l)\Gamma(1+m)\Gamma(1-\sigma-\frac{1}{2}D)}{\Gamma(1+\sigma)\Gamma(1+j+m+\frac{1}{2}D)\Gamma(1+i-\sigma)\Gamma(1-m-\frac{1}{2}D-\sigma)\Gamma(1+l+m+\frac{1}{2}D)} \times \sum_{n_2,n_3=0}^{\infty} \frac{(-1)^{n_2+n_3}(-\sigma|n_2)(-i+\sigma|n_3)}{n_2!n_3!(1+j+m+\frac{1}{2}D|n_3-n_2)(1-m-\frac{1}{2}D-\sigma|n_2-n_3)}, \tag{A.6}$$

$$S_9 = \frac{(-\pi)^D (p^2)^\sigma \Gamma(1+i)\Gamma(1+j)\Gamma(1+l)\Gamma(1+m)\Gamma(1-\sigma-\frac{1}{2}D)}{\Gamma(1+\sigma)\Gamma(1+i+l+\frac{1}{2}D)\Gamma(1+j-\sigma)\Gamma(1-l-\frac{1}{2}D-\sigma)\Gamma(1+l+m+\frac{1}{2}D)} \times \sum_{n_1,n_6=0}^{\infty} \frac{(-1)^{n_1+n_6}(-\sigma|n_1)(-j+\sigma|n_6)}{n_1!n_6!(1+i+l+\frac{1}{2}D|n_6-n_1)(1-l-\frac{1}{2}D-\sigma|n_1-n_6)}, \tag{A.7}$$

$$S_{10} = \frac{(-\pi)^D (p^2)^\sigma \Gamma(1+i)\Gamma(1+j)\Gamma(1+l)\Gamma(1+m)\Gamma(1-\sigma-\frac{1}{2}D)}{\Gamma(1+\sigma)\Gamma(1-m-\frac{1}{2}D)\Gamma(1+j-\sigma)\Gamma(1-j-l-\frac{1}{2}D)} \times \frac{1}{\Gamma(1+l+m+\frac{1}{2}D)} \sum_{n_1,n_5=0}^{\infty} \frac{(-\sigma|n_1)(-j+\sigma|n_5)(\frac{1}{2}D+m|n_1+n_5)}{n_1!n_5!(1-j-l-\frac{1}{2}D|n_1+n_5)}, \tag{A.8}$$

$$S_{11} = \frac{(-\pi)^D (p^2)^\sigma \Gamma(1+i)\Gamma(1+j)\Gamma(1+l)\Gamma(1+m)\Gamma(1-\sigma-\frac{1}{2}D)}{\Gamma(1+\sigma)\Gamma(1+j+l+\frac{1}{2}D)\Gamma(1+i-\sigma)\Gamma(1-l-\frac{1}{2}D-\sigma)\Gamma(1+l+m+\frac{1}{2}D)} \times \sum_{n_1,n_4=0}^{\infty} \frac{(-1)^{n_1+n_4}(-\sigma|n_1)(-i+\sigma|n_4)}{n_1!n_4!(1+j+l+\frac{1}{2}D|n_4-n_1)(1-l-\frac{1}{2}D-\sigma|n_1-n_4)}, \tag{A.9}$$

$$S_{12} = \frac{(-\pi)^D (p^2)^\sigma \Gamma(1+i)\Gamma(1+j)\Gamma(1+l)\Gamma(1+m)\Gamma(1-\sigma-\frac{1}{2}D)}{\Gamma(1+\sigma)\Gamma(1-i-l-\frac{1}{2}D)\Gamma(1+i-\sigma)\Gamma(1-m-\frac{1}{2}D)} \times \frac{1}{\Gamma(1+l+m+\frac{1}{2}D)} \sum_{n_1,n_2=0}^{\infty} \frac{(-\sigma|n_1)(-i+\sigma|n_3)(\frac{1}{2}D+m|n_1+n_3)}{n_1!n_2!(1-i-l-\frac{1}{2}D|n_1+n_3)}, \tag{A.10}$$

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